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Abstract

We present a study of the incremental projection method to solve incompressible unsteady Stokes equations based on a low degree nonconforming finite element approximation in space, with, in particular, a piecewise constant approximation for the pressure. The numerical method falls in the class of algebraic projection methods. We provide an error analysis in the case of Dirichlet boundary conditions, which confirms that the splitting error is second order in time. In addition, we show that pressure artificial boundary conditions are present in the discrete pressure elliptic operator, even if this operator is obtained by a splitting performed at the discrete level; however, these boundary conditions are imposed in the finite volume (weak) sense and the optimal order of approximation in space is still achieved, even for open boundary conditions.

MCS Classification : 76D05, 35Q30, 76F65, 76D03

Key-words : incompressible flows, unsteady Stokes problem, projection methods, Rannacher-Turek finite elements

1 Introduction

We consider the time-dependent incompressible Stokes equations, posed on a finite time interval $(0,T)$ and in an open, connected, bounded domain Ω in $\mathbb{R}^d$ ($d=2$, or 3), which is supposed to be polygonal ($d=2$) or polyhedral ($d=3$) for the sake of simplicity. The system under consideration reads:

$$\partial_t \boldsymbol{u} - \triangle \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega, \tag{1}$$

where $\boldsymbol{u}$ stands for the (vector-valued) velocity, p for the (scalar) pressure, and $\boldsymbol{f}$ for a (vector-valued regular) known forcing term. The boundary Γ of Ω is supposed to be split in $\Gamma = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \neq \emptyset$, and the velocity is prescribed over Γ_D while Neumann boundary conditions are imposed over Γ_N:

$$\boldsymbol{u} = \boldsymbol{u}_{\Gamma_D} \text{ on } (0,T)\times\Gamma_D, \quad -p\boldsymbol{n} + \boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{f}_N \text{ in } (0,T)\times\Gamma_N. \tag{2}$$

This system must be supplemented by the initial condition $\boldsymbol{u} = \boldsymbol{u}_0$ on Ω, for $t=0$. The vector fields $\boldsymbol{u}_{\Gamma_D}$, $\boldsymbol{f}_N$ and $\boldsymbol{u}_0$ are supposed to be given and regular.

We present in this paper a discretization of System (1) with the nonconforming low-degree Rannacher-Turek element [6]. The time discretization is performed by an incremental projection method [1, 8]. Since the pressure is approximated by piecewise constant functions, the projection step must be left as a Darcy system. We thus choose to use a lumped discretization for the time -derivative terms, which allows us to obtain the elliptic problem for the pressure by an explicit algebraic process.

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Our results are twofold. First, we are able to lay down the scheme in a variational setting, with mesh-dependent inner-products, operators and norms, which allows us to adapt for the problem at hand the error analyses performed in the semi-discrete time setting [7, 3] or for conforming elements [2]; we thus obtain, for homogeneous Dirichlet boundary conditions, a second-order estimate (with respect to the time step) for the splitting error. Second, we derive an explicit expression for discrete elliptic operator applied to the pressure increment in the projection step. This construction brings some new element to the rather controversial issue (in the framework of algebraic methods) of artificial pressure boundary conditions (see [4] and references therein): indeed, we show that we obtain a finite-volume-like discretization of the Laplace operator, with the expected boundary conditions, namely homogeneous Neuman and Dirichlet boundary conditions on Γ_D and Γ_N respectively; however, since, as usual for finite volumes, these boundary conditions are only enforced in a weak sense, their influence is observed to vanish when the time step goes to zero, and we recover optimal convergence rates with respect to the size of the mesh, even in the L^∞ norm for the pressure in the case of open boundary conditions.

This paper is organized as follows. We first describe the scheme (section 2), then we give the expression of the elliptic pressure operator (section 3), we provide error bounds (section 4), and finally, describe some numerical tests which substantiate our analyses (section 5).

2 Discretization

2.1 The standard time-discrete projection algorithm

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which is supposed uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the constant time step. In a time semi-discrete setting, denoting by $\boldsymbol{u}^0$ initial guess for the velocity, the usual incremental projection scheme reads, for $0 \leq n < N$:

1 - Solve for $\tilde{\boldsymbol{u}}^{n+1}$

$$\frac{1}{\delta t}(\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n) - \triangle\tilde{\boldsymbol{u}}^{n+1} + \boldsymbol{\nabla} p^n = \boldsymbol{f}^{n+1}, \tag{3}$$

with the boundary conditions $\tilde{\boldsymbol{u}}^{n+1} = \boldsymbol{u}_{\Gamma_D}^{n+1}$ on Γ_D and $\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{n} - p^n \boldsymbol{n} = \boldsymbol{f}_N^{n+1}$ on Γ_N.

2 - Solve for p^{n+1} and $\boldsymbol{u}^{n+1}$

$$\frac{1}{\delta t}(\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}) + \boldsymbol{\nabla}(p^{n+1} - p^n) = 0, \quad \mathrm{div}\boldsymbol{u}^{n+1} = 0, \tag{4}$$

with the boundary conditions $\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{n}$ on Γ_D and $p^{n+1} = p^n$ on Γ_N.

Usually, for the solution of Step 2, the two equations are combined (taking the divergence of the first equality and substracting to the second one) to obtain an elliptic problem for the pressure, which reads in the time semi-discrete setting:

$$-\triangle(p^{n+1} - p^n) = -\frac{1}{\delta t}\mathrm{div}\tilde{\boldsymbol{u}}^{n+1} \text{ in } \Omega, \quad \boldsymbol{\nabla}(p^{n+1} - p^n).\boldsymbol{n} = 0 \text{ on } \Gamma_D, \quad p^{n+1} = p^n \text{ on } \Gamma_N. \tag{5}$$

Boundary conditions of (3), (4) and (5) are in a sense consistent with the strong formulation of the problem, since they enforce the fact that $\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}_{\Gamma_D}^{n+1} \cdot \boldsymbol{n}$ on Γ_D and $\boldsymbol{\nabla}\tilde{\boldsymbol{u}}^{n+1} \cdot \boldsymbol{n} - p^{n+1}\boldsymbol{n} = \boldsymbol{f}_N^{n+1}$ on Γ_N. In addition, they have for consequence that $\boldsymbol{u}^{k+1}$ is the orthogonal L^2 projection of $\tilde{\boldsymbol{u}}^{k+1}$ on

$$H = \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^d, \ \mathrm{div}\boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{u}_{\Gamma_D}^{n+1} \cdot \boldsymbol{n} \text{ on } \Gamma_D\}.$$

However, the tangent components of the end-of-step velocity do not satisfy Dirichlet boundary conditions, and spurious boundary conditions are enforced to the pressure on the whole boundary.

2.2 The full discrete scheme

Let $\mathcal{M}$ be a decomposition of the domain Ω into quadrangles ($d=2$) or hexahedra ($d=3$), supposed to be regular in the usual sense of the finite element literature. We denote by $\mathcal{E}$ the set of all faces σ of the mesh; by $\mathcal{E}_{ext}$ the set of faces included in the boundary of Ω, by $\mathcal{E}_{int}$ the set of internal faces (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) and by $\mathcal{E}(K)$ the faces of a particular cell $K \in \mathcal{M}$. The internal face separating the neighbour cells K and L is denoted by $\sigma = K|L$. For each cell $K \in \mathcal{M}$ and each face $\sigma \in \mathcal{E}(K)$, $\boldsymbol{n}_{K,\sigma}$ stand for the normal vector to σ outward K. By $|K|$ and $|\sigma|$ we denote the measure, respectively, of the control volume K and of the face σ.

The velocity and the pressure are discretized using the so-called Rannacher-Turek finite element [6]. The approximation for the velocity is thus non-conforming: the space X_h is composed of discrete functions which are discontinuous through an edge, but the jump of their integral is imposed to be zero; the degrees of freedom are located at the center of the edges of the mesh, and we choose the version of the element where they represent the average of the velocity through an edge. The set of degrees of freedom thus reads:

$$\{\boldsymbol{u}_{\sigma,i},\ \sigma \in \mathcal{E},\ 1 \leq i \leq d\}.$$

We denote by $\boldsymbol{\varphi}_\sigma^{(i)}$ the vector shape function associated to $\boldsymbol{u}_{\sigma,i}$, which, by definition, reads $\boldsymbol{\varphi}_\sigma^{(i)} = \varphi_\sigma\, \boldsymbol{e}^{(i)}$, where φ_σ is the Rannacher-Turek scalar shape function and $\boldsymbol{e}^{(i)}$ is the $i^{\rm th}$ vector of the canonical basis of $\mathbb{R}^d$, and we define $\boldsymbol{u}_\sigma$ by $\boldsymbol{u}_\sigma = \sum_{i=1}^d \boldsymbol{u}_{\sigma,i}\, \boldsymbol{e}^{(i)}$. With these definitions, we have the identity:

$$\boldsymbol{u} = \sum_{\sigma\in\mathcal{E}} \sum_{i=1}^{d} \boldsymbol{u}_{\sigma,i}\ \boldsymbol{\varphi}_\sigma^{(i)}(\boldsymbol{x}) = \sum_{\sigma\in\mathcal{E}} \boldsymbol{u}_\sigma\ \varphi_\sigma(\boldsymbol{x}), \qquad \text{for a.e. } \boldsymbol{x} \in \Omega.$$

Let $\mathcal{E}_D \subset \mathcal{E}_{\rm ext}$ be the set of edges where the velocity is prescribed, let say to $\boldsymbol{u} = \boldsymbol{u}_D$. Then, as usual, these Dirichlet boundary conditions are built-in in the definition of the discrete space:

$$\forall \sigma \in \mathcal{E}_D,\ \text{for } 1 \leq i \leq d, \qquad \boldsymbol{u}_{\sigma,i} = \frac{1}{|\sigma|} \int_\sigma \boldsymbol{u}_{D,i}, \tag{6}$$

where $\boldsymbol{u}_{D,i}$ stands for the $i^{\rm th}$ component of $\boldsymbol{u}_D$. For $\boldsymbol{v} \in X_h$, we denote by $\boldsymbol{\nabla}_h \boldsymbol{v}$ and ${\rm div}_h \boldsymbol{v}$ the functions of ${\rm L}^2(\Omega)^{d\times d}$ and ${\rm L}^2(\Omega)$ respectively equal to $\boldsymbol{\nabla v}$ and ${\rm div}\boldsymbol{v}$ almost everywhere in Ω.

The pressure is piecewise constant, and its degrees of freedom are p_K for any cell $K \in \mathcal{M}$. We denote by M_h the discrete pressure space.

To obtain our fractional step algorithm, as in the usual incremental scheme presented in the previous section, we split the resolution in two steps: the beginning-of-step velocity $\boldsymbol{u}^n \in X_h$ and pressure $p^n \in M_h$ being known, we first perform a prediction step to obtain a tentative (non divergence free) velocity $\tilde{\boldsymbol{u}}^{n+1} \in X_h$, then we compute the end-of-step pressure $p^{n+1} \in M_h$ and (divergence free) velocity $\boldsymbol{u}^{n+1} \in X_h$ in a second step. We obtain, for $0 \leq n < N$:

1 - Velocity prediction step:

We seek for $\tilde{\boldsymbol{u}}^{n+1} \in X_h$ such that (6) holds with $\boldsymbol{u}_D = \boldsymbol{u}_{\Gamma_D}$ and, for any face $\sigma \in \mathcal{E} \setminus \mathcal{E}_D$, any integer i in $\{1, \cdots, d\}$:

$$\frac{|D_\sigma|}{\delta t} \left[\tilde{\boldsymbol{u}}_{\sigma,i}^{n+1} - \boldsymbol{u}_{\sigma,i}^n\right] + \int_\Omega \boldsymbol{\nabla}_h \tilde{\boldsymbol{u}}^{n+1} : \boldsymbol{\nabla}_h \boldsymbol{\varphi}_\sigma^{(i)} - \int_\Omega p^n\ {\rm div}_h \boldsymbol{\varphi}_\sigma^{(i)} = \int_\Omega \boldsymbol{f}^{n+1} \cdot \boldsymbol{\varphi}_\sigma^{(i)} + \int_{\Gamma_N} \boldsymbol{f}_N^{n+1} \cdot \boldsymbol{\varphi}_\sigma^{(i)},$$

where $|D_\sigma| = \displaystyle\int_\Omega \varphi_\sigma$.

The pressure gradient term in this relation may equivalently be written, for $1 \leq i \leq d$:

$$\begin{aligned} &\forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L && \int_\Omega p^n\ {\rm div}_h \boldsymbol{\varphi}_\sigma^{(i)} = |\sigma|\ (p_K^n - p_L^n)\ \boldsymbol{n}_{K,\sigma}, \\ &\forall \sigma \in \mathcal{E}_{\rm ext} \setminus \mathcal{E}_D,\ \sigma \in \mathcal{E}(K), && \int_\Omega p^n\ {\rm div}_h \boldsymbol{\varphi}_\sigma^{(i)} = |\sigma|\ p_K^n\ \boldsymbol{n}_{K,\sigma}. \end{aligned} \tag{7}$$

2 - Velocity projection step:

We seek $\boldsymbol{u}^{n+1} \in X_h$ and p^{n+1} in M_h such that (6) holds with $\boldsymbol{u}_D = \boldsymbol{u}_{\Gamma_D}$ and:

$$(8) \qquad \begin{aligned} &\forall \sigma \in \mathcal{E} \setminus \mathcal{E}_D, \text{ for } 1 \le i \le d, && \frac{|D_\sigma|}{\delta t} \left[\boldsymbol{u}^{n+1}_{\sigma,i} - \tilde{\boldsymbol{u}}^{n+1}_{\sigma,i} \right] - \int_\Omega (p^{n+1} - p^n) \, \mathrm{div}_h \boldsymbol{\varphi}^{(i)}_\sigma = 0, \\ &\forall K \in \mathcal{M}, && \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \boldsymbol{u}^{n+1}_\sigma \cdot \boldsymbol{n}_{K,\sigma} = 0. \end{aligned}$$

At first glance, comparing to the semi-discrete version of the incremental projection algorithm, it may be puzzling that the whole set of Dirichlet boundary conditions (6) be enforced to the end of step velocity. In fact, the expression of the discrete gradient (7) shows that, for the specific discretization considered here, the discrete pressure gradient on a face σ is colinear to its normal vector, so the velocities tangent to the faces (and thus to the boundary of the domain) are anyway left unchanged by the correction step (*i.e.* may them be prescribed or not).

3 The discrete pressure elliptic problem . . . and pressure artificial boundary conditions

Since the discrete pressure elliptic problem is not posed explicitely (as at the continuous level), neither are the associated boundary conditions for the pressure increment. We are going to show that these boundary conditions are however recovered when computing the discrete operator.

To this purpose, let us multiply the first equation of the velocity projection step (8) by $\dfrac{1}{|D_\sigma|} \, |\sigma| \, \boldsymbol{n}^{(i)}_{K,\sigma}$ and sum up the equations obtained for $1 \le i \le d$ and $\sigma \in \mathcal{E}(K)$. We get, for any $K \in \mathcal{M}$:

$$\sum_{\sigma \in \mathcal{E}_{\mathrm{int}}, \sigma = K|L} \frac{|\sigma|^2}{|D_\sigma|} \, \left[\phi^{n+1}_K - \phi^{n+1}_L \right] + \sum_{\sigma \in (\mathcal{E} \setminus \mathcal{E}_D) \cap \mathcal{E}(K)} \frac{|\sigma|^2}{|D_\sigma|} \, \phi^{n+1}_K = \frac{1}{\delta t} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \tilde{\boldsymbol{u}}^{n+1}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$$

where we have set $\phi_K = p^{n+1}_K - p^n_K, \ \forall K \in \mathcal{M}$. We recognize in the left-hand side of this relation a finite-volume like approximation of the Laplace operator, however unconsistent, since, on a uniform mesh, it can easily be seen that the coefficient $|\sigma|^2/|D_\sigma|$ is d times greater than in finite-volume scheme, this being probably related to the fact that the Rannacher-Turek elements are known to provide an unconsistent approximation of the Darcy problem. The expected artificial boundary conditions (*i.e.* those of the time semi-discrete algorithm), namely homogeneous Neumann boundary conditions on any $\sigma \in \mathcal{E}_D$ and homogeneous Dirichlet boundary conditions on any $\sigma \in \mathcal{E}_{\mathrm{ext}} \setminus \mathcal{E}_D$, are built-in in this operator. However, on Γ_N, boundary conditions are imposed in a weaker sense than in conformal approximations where pressure degrees of freedom lie on the boundary (in this latter case, pressure increments on the boundary are, usually, set exactly to zero). We may thus expect this boundary condition to be relaxed when the time step goes to zero; this behaviour is indeed observed in numerical experiments.

4 Discrete variational formulation and error estimates

We suppose in this section that the velocity is prescribed to zero on the whole boundary, *i.e.* $\Gamma_N = \emptyset$ and $\boldsymbol{u}_{\Gamma_D} = \boldsymbol{u}_{\Gamma_D} = 0$. We consider the implicit scheme as a reference scheme, denote by $(\overline{\boldsymbol{u}}^n, \overline{p}^n) \in X_h \times M_h$, $1 \le n \le N$, its solution, and define the splitting errors by:

$$\boldsymbol{e}^n = \boldsymbol{u}^n - \overline{\boldsymbol{u}}^n, \quad \widetilde{\boldsymbol{e}}^n = \tilde{\boldsymbol{u}}^n - \overline{\boldsymbol{u}}^n, \quad \text{and} \quad \epsilon^n = p^n - \overline{p}^n.$$

By taking the difference of the equations of both schemes, summing over $\sigma \in \mathcal{E} \setminus \mathcal{E}_D$ and $1 \leq i \leq d$, we deduce the following discrete variational formulation:

$$
(9) \quad \begin{aligned}
&\boldsymbol{e}^{n+1} \in X_h,\ \widetilde{\boldsymbol{e}}^{n+1} \in X_h, \epsilon^{n+1} \in M_h,\ \text{and}\ , \forall (\boldsymbol{v}, q) \in X_h \times M_h,\\
&\sum_{\sigma \in \mathcal{E}} \frac{|D_\sigma|}{\delta t} \left[\widetilde{\boldsymbol{e}}_\sigma^{n+1} - \boldsymbol{e}_\sigma^n \right] \cdot \boldsymbol{v}_\sigma + \int_\Omega \boldsymbol{\nabla}_h \widetilde{\boldsymbol{e}}^{n+1} : \boldsymbol{\nabla}_h \boldsymbol{v} - \int_\Omega \epsilon^n \ \mathrm{div}_h \boldsymbol{v} = \int_\Omega (\overline{p}^{n+1} - \overline{p}^n)\ \mathrm{div}_h \boldsymbol{v},\\
&\sum_{\sigma \in \mathcal{E}} \frac{|D_\sigma|}{\delta t} \left[\boldsymbol{e}_\sigma^{n+1} - \widetilde{\boldsymbol{e}}_\sigma^{n+1} \right] \cdot \boldsymbol{v}_\sigma \int_\Omega (\epsilon^{n+1} - \epsilon^n)\ \mathrm{div}_h \boldsymbol{v} = \int_\Omega (\overline{p}^{n+1} - \overline{p}^n)\ \mathrm{div}_h \boldsymbol{v},\\
&\int_\Omega q\ \mathrm{div}_h \boldsymbol{e}^{n+1} = 0,
\end{aligned}
$$

where we have supposed that all the functions of X_h vanish at the boundary.
We now define the following discrete inner products, norm and semi-norm:

$$
\begin{aligned}
&\forall\ (\boldsymbol{u}, \boldsymbol{v}) \in X_h^2, \qquad (\boldsymbol{u}, \boldsymbol{v})_h = \sum_{\sigma \in \mathcal{E}} |D_\sigma|\ \boldsymbol{u}_\sigma \cdot \boldsymbol{v}_\sigma, \qquad \|\boldsymbol{u}\|_{0,h}^2\ = (\boldsymbol{u}, \boldsymbol{u})_h\\
&\forall\ (p, q) \in M_h^2, \qquad \langle p, q \rangle_h = \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ (\sigma = K|L)}} \frac{|\sigma|^2}{|D_\sigma|} (p_K - p_L)\ (q_K - q_L) \qquad |p|_{1,h}^2\ = \langle p, p \rangle_h
\end{aligned}
$$

With these notations, we recover the structure used in pressure correction schemes analyses [3], and so, with some technical adaptations (in particular, the proof of some properties of the discrete inverse Stokes problem, assuming that the continuous Stokes problem is regularizing, which, in our case, reduces to the fact that the domain Ω is convex), we are able to derive the known second order for the velocity and first order for the pressure error estimates. Under the assumptions of regularity on the solution of the implicit scheme and the regularizing effect of the Stokes problem, we prove the following results in the case of homogeneous Dirichlet boundary conditions on $\partial\Omega$:

Theorem 4.1 *Assume that the implicit problem is regular, in the sense that there exists $C > 0$ such that, for $1 \leq n \leq N-1$:*

$$
|\overline{p}^{n+1} - 2\overline{p}^n + \overline{p}^{n-1}|_{1,h}\ \leq C\delta t^2, \qquad \sum_{k=1}^n |\overline{p}^{k+1} - \overline{p}^k|_{1,h}^2\ \leq C\delta t,
$$

which, basically, means that the second derivative with respect to time of the pressure gradient is uniformly bounded. Then there exists $c > 0$ such that, for $1 \leq n \leq N$:

$$
\left(\sum_{k=0}^n \delta t\ \|\boldsymbol{e}_h^k\|_{0,h}^2 \right)^{1/2} + \left(\sum_{k=0}^n \delta t\ \|\widetilde{\boldsymbol{e}}_h^k\|_{0,h}^2 \right)^{1/2} + \delta t \left(\sum_{k=0}^n \delta t\ \|\epsilon_h^k\|_0{}^2 \right)^{1/2} \leq c\delta t^2.
$$

5 Numerical tests

Various numerical tests have been performed, and they confirm the error analysis. For short, we only present here a problem with open boundary conditions.
The computational domain Ω is the unit square $[0,1]^2$ with Γ_N equal to the vertical left side (and so $\Gamma_D = \partial\Omega \backslash \Gamma_N$ is equal to the three other sides). We calculate the forcing term $\boldsymbol{f}$ such that the exact velocity and pressure fields, $\boldsymbol{u}_e$ and p_e, be:

$$
\boldsymbol{u}_e(x,y,t) = \begin{bmatrix} \sin(x)\sin(y+t) \\ \cos(x)\cos(y+t) \end{bmatrix}, \quad p_e(x,y,t) = \cos(x)\sin(y+t).
$$

The initial and boundary conditions are chosen in order to match the solution, so, for instance, we obtain on Γ_N:

$$
\boldsymbol{\nabla u} \cdot \boldsymbol{n} - p\boldsymbol{n} = 0.
$$

We plot in Figure 1 the numerical error as a function of the time step, measured in L^2-norm and calculated at a fixed time, for 20×20, 40×40 and 80×80 structured uniform meshes. The errors first decrease with the time step, then a plateau is reached, which corresponds to the residual error in space. The order of convergence in time is close to 2 for the velocity (slope of the left curve) and 1 for the pressure (right); this may be surprising, since we use an only first order backward Euler scheme, but, consistently with the error analysis given above, is explained by the fact that the error is essentially due to the splitting. On the plateau, we observe a second order convergence in space for the velocity and first order for the pressure, that is the optimal order of convergence with our (low-degree) approximation; by comparison, in [5], the authors observed for a Taylor-Hood ($i.e.P_2 - P_1$) approximation only a first order convergence for the velocity and 1/2 for the pressure, so, for the velocity, the present computations become more accurate already for the 80×80 mesh.

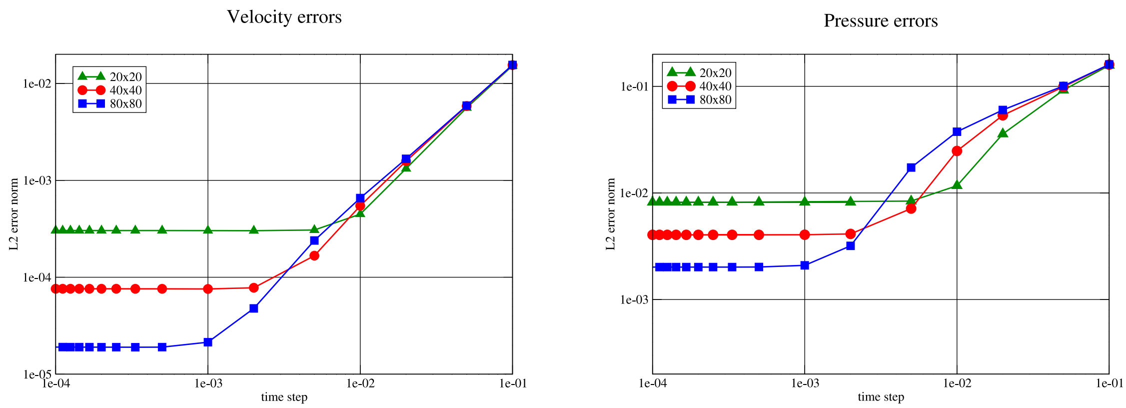

Figure 1: Errors of approximation (L^2 norm) as a function of δt.

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